Cambridge International AS & A Level

Mathematics 9709/32

Paper 3 Pure Mathematics 3 October/November 2024

Question No(10)

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A balloon in the shape of a sphere has volume V and radius r. Air is pumped into the balloon at a constant rate of 40π starting when time t = 0 and r =0. At the same time, air begins to flow out of the balloon at a rate of $0.80\ \pi r$. The balloon remains a sphere at all times.

(a) Show that r and t satisfy the differential equation

$$\frac{dr}{dt} = \frac{50-r}{5r^2}$$

(b) Find the quotient and remainder when $5r^2$ is divided by $50 - r$.

(c) Solve the differential equation in part (a), obtaining an expression for t in terms of r.

(d) Find the value of t when the radius of the balloon is 12.

Solution:

(a)

Volume of sphere

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = \frac{4}{3}\pi(3r^2)$$

$$= 4\pi r^2$$

Now

$$\frac{dV}{dt} = \frac{4}{3}\left(3\pi r^2 \frac{dr}{dt}\right)$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \longrightarrow (1)$$

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As given in the statement of question rate of change of volume

$$\frac{dv}{dt} = 40\pi - 0.8\pi r$$ (As -0.8π flow out of the balloon)

From equation (1)

$$\frac{dr}{dt} = \frac{dv/dt}{4\pi r^2}$$

$$= \frac{40\pi - 0.8\pi r}{4\pi r^2} \qquad \because \frac{dv}{dt} = 40\pi - 0.8\pi r$$

$$\frac{dr}{dt} = \frac{40\pi - \frac{8}{10}\pi r}{4\pi r^2}$$

$$= \frac{400\pi - 8\pi r}{40\pi r^2}$$

$$= \frac{8\pi(50-r)}{40\pi r^2}$$

$$\frac{dr}{dt} = \frac{50-r}{5r^2}$$

(b) Find the quotient and remainder when $5r^2$ is divided by $50-r$

Solution

DATE :-

(29)

$$\begin{array}{r|l} & -5r-250 \\ \hline 50-r & 5r^2 \\ & \underline{-5r^2 \mp 250r} \\ & \quad 250r \\ & \quad \underline{-250r \mp 12500} \\ & \qquad 12500 \end{array}$$

quotient $= -5r-250$

remainder $= 12500$

(c)

solve the differential equation in part (a), obtaining an expression for t in terms of r.

solution

From (a)

$$\frac{dr}{dt} = \frac{50-r}{5r^2}$$

separate the variables

$$dt = \frac{5r^2}{50-r}\,dr$$

$$dt = \left(-5r-250+\frac{12500}{50-r}\right)dr$$

From part(b)

(30)

integrate

$$\int dt = \int \left(-5r - 250 + \frac{12500}{50-r}\right) dr$$

$$t = -\frac{5r^2}{2} - 250r - 12500 \ln(50-r) + C \rightarrow (1)$$

$$\int \frac{dr}{r} = \ln r$$

where c is constant of integration.

As given, at $t=0$, $r=0$ above equation becomes

$$0 = 0 - 0 - 12500 \ln 50 + c$$

$$c = 12500 \ln 50$$

put value of $C = 12500 \ln 50$ in (1)

$$t = -\frac{5}{2}r^2 - 250r - 12500 \ln(50-r) + 12500 \ln 50$$

(d) Find the value of t when the radius of the balloon is 12

Solution

From part (C)

$$t = -\frac{5}{2}r^2 - 250r - 12500 \ln(50-r) + 12500 \ln 50$$

at $r=12$

$$t = -\frac{5}{2}(12)^2 - 250(12) - 12500 \ln(50-12) + 12500 \ln 50$$

$$t = 70.5$$