Cambridge International AS & A Level

Mathematics 9709/32

Paper 3 Pure Mathematics 3 October/November 2024

Question No(11)

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Let $f(x) = \frac{2e^{2x}}{e^{2x}-3e^{x}+2}$

(a) Find $f'(x)$ and hence find the exact coordinates of the stationary point of the curve with equation $y = f(x)$

(b) Use the substitution $u = e^{x}$ and partial fractions to find the exact value of $\int_{ln3}^{ln5} f(x)dx$.
Give your answer in the form lna, where a is a rational number in its simplest form.

Solution:

$$f(x) = \frac{2e^{2x}}{e^{2x}-3e^{x}+2}$$

Formula

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$f'(x) = \frac{(e^{2x}-3e^{x}+2)\frac{d}{dx}(2e^{2x}) - 2e^{2x}\frac{d}{dx}(e^{2x}-3e^{x}+2)}{(e^{2x}-3e^{x}+2)^2}$$

(32)

$$= \frac{(e^{2x}-3e^{x}+2)(4e^{2x}) - 2e^{2x}(2e^{2x}-3e^{x}+0)}{(e^{2x}-3e^{x}+2)^2}$$

$$= \frac{2e^{2x}\left[2(e^{2x}-3e^{x}+2) - (2e^{2x}-3e^{x})\right]}{(e^{2x}-3e^{x}+2)^2}$$

For the stationary point $f'(x)=0$

$$2e^{2x}-6e^{x}+4-2e^{2x}+3e^{x}=0$$

$$-3e^{x}+4=0$$

$$-3e^{x}=-4$$

$$e^{x}=4/3$$

$$\ln e^{x} = \ln(4/3)$$

$$x \ln e = \ln(4/3)$$

$$x = \ln(4/3) \qquad \because \ln e = 1$$

As $y = f(x)$

$$y = \frac{2e^{2x}}{e^{2x}-3e^{x}+2} \qquad \because f(x) = \frac{2e^{2x}}{e^{2x}-3e^{x}+2}$$

put value of $x = \ln(4/3)$

$$y = \frac{2e^{2(\ln 4/3)}}{e^{2(\ln 4/3)} - 3e^{\ln 4/3}+2}$$

DATE :-

(33)

$$= \frac{2\,(4/3)^2}{(4/3)^2 - 3(4/3) + 2}$$

$\because e^{\ln x} = x$

$$= \frac{2\left(16/9\right)}{\frac{16}{9} - \frac{12}{3} + 2}$$

$$= \frac{32/9}{\frac{16 - 36 + 8}{9}}$$

$$= \frac{32}{-2}$$

$y = -16$

So The stationary point is

$$\left(\ln 4/3,\ -16\right)$$

(b) Use The substitution $u = e^x$ and partial fractions to find The exact value of $\int_{\ln 3}^{\ln 5} f(x)\,dx$

Give your answer in The form $\ln a$, where a is a rational number in its simplest form.

(34)

solution

$$\int_{\ln 3}^{\ln 5} f(x)\,dx = \int_{\ln 3}^{\ln 5} \frac{2e^{2x}}{e^{2x}-3e^{x}+2}\,dx \rightarrow (1)$$

According to given condition put

$$u = e^{x}$$

$$\frac{du}{dx} = e^{x}$$

$$\frac{du}{dx} = e^{x}$$

$$\frac{du}{e^{x}} = dx$$

$$\frac{du}{u} = dx \qquad \because e^{x} = u$$

NOW for limit according to variable u

As $u = e^{x}$

$u = e^{\ln 5}$ $\qquad \because x = e^{x}$ upper limit

$u = 5$

Similarly

$u = e^{x}$

$= e^{\ln 3}$ $\qquad$ lower limit

$u = 3$

DATE :-

35

NOW equation 1 becomes

$$= \int_3^5 \frac{2u^2}{u^2-3u+2}\left(\frac{du}{u}\right)$$

$$= \int_3^5 \frac{2u\,du}{u^2-3u+2}$$

$$= \int_3^5 \frac{2u}{u^2-2u-u+2}\,du \quad \text{by factorization}$$

$$= \int_3^5 \frac{2u}{u(u-2)-1(u-2)}\,du$$

$$= \int_3^5 \frac{2u}{(u-2)(u-1)}\,du \longrightarrow 2$$

$$\frac{2u}{(u-2)(u-1)} = \frac{A}{u-2} + \frac{B}{u-1} \longrightarrow 3 \quad \text{By partial fraction}$$

$$2u = A(u-1) + B(u-2) \longrightarrow 4$$

put $u-2=0$

$u=2$ in equation (4)

$A=4$

36

put $u-1=0 \Rightarrow u=1$ in equation (4)

$2(1) = -B$

$B = -2$

Now Equation (3) after putting values of A and B

$$\frac{2u}{(u-2)(u-1)} = \frac{4}{u-2} - \frac{2}{u-1}$$

put $\frac{2u}{(u-2)(u-1)}$ value in equation (2)

$$= \int_3^5 \left(\frac{4}{u-2} - \frac{2}{u-1}\right) du$$

$$= \left| 4\ln(u-2) - 2\ln(u-1) \right|_3^5$$

$$= [4\ln(5-2) - 2\ln(5-1)] - [4\ln(3-2) - 2\ln(3-1)]$$

$$= 4\ln 3 - 2\ln 4 - [4\ln 1 - 2\ln 2]$$

$$= \ln 3^4 - \ln 4^2 - 0 + 2\ln 2$$ ∵ $\ln 1 = 0$, ln property $n\ln x = \ln x^n$

$$= \ln 81 - \ln 16 + \ln 2^2$$

$$= \ln 81 - \ln 16 + \ln 4$$

$$= \ln\left(\frac{81 \times 4}{16}\right)$$ ∵ $\ln\left(\frac{xy}{z}\right) = \ln(xy) - \ln z$

$$= \ln\left(\frac{81}{4}\right)$$