Cambridge International AS & A Level

Mathematics 9709/12

Paper 1 Pure Mathematics 1 **May/June 2024**

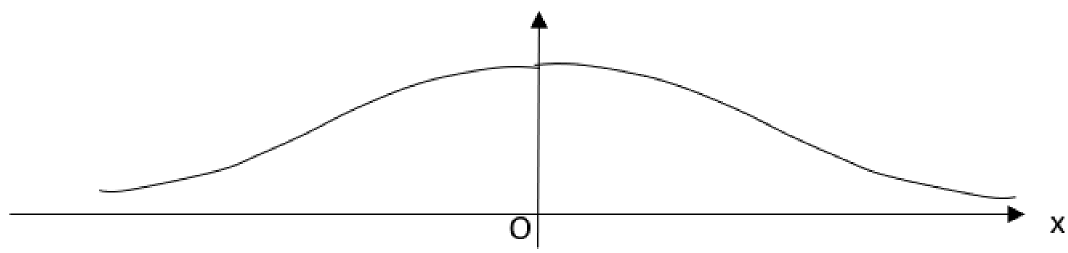
Question No(4)

Question No (4)

The function f is defined as follows:

$$f(x) = \sqrt{x} - 1 \; for \; x > 1.$$

$(a)\ Find\ an\ expression\ for\ f^{-1}(x)$



The diagram shows the graph of $y = g(x)\ \ where\ g(x) = \frac{1}{x^2+2}\ \ for\ x \in R.$

(b) $State\ the\ range\ of\ g\ and\ explain\ whether\ g^{-1}\ exists.$

$The\ function\ h\ is\ defined\ by\ h(x) = \dfrac{1}{x^2+2}\ \ for\ x\ \geq 0$

$(c)\ \ Solve\ the\ equation\ hf(x) = f\left(\frac{25}{16}\right).$

$Give\ your\ answer\ in\ the\ form\ a + b\sqrt{c}\ \ , where\ a, b\ and\ c\ are\ integers.$

Solution:

(a)

$f(x) = \sqrt{x} - 1$

$y = \sqrt{x} - 1$ $\qquad$ $f(x) = y$

$\qquad\qquad\qquad\quad$ $x = f^{-1}(y)$

$\sqrt{x} = y + 1$

$x = (y+1)^2$

$f^{-1}(y) = (y+1)^2$

$f^{-1}(x) = (x+1)^2$ $\qquad$ change y with x

(b)

As $g(x) = \frac{1}{x^2+2}$ for all $x \in R$

Due to +ve 2 in denominator each value in range will be ~~+ve~~ less than $\frac{1}{2}$ and +ve as well due to x^2.

So the range is

$$0 < g(x) \leq \frac{1}{2}$$

Now we check the inverse.

$$g(x) = \frac{1}{x^2+2}$$

$$y = \frac{1}{x^2+2} \qquad \because y = g(x)$$

$$x^2 + 2 = \frac{1}{y}$$

$$x^2 = \frac{1}{y} - 2$$

$$x = \pm\sqrt{\frac{1}{y} - 2}$$

$$g^{-1}(y) = \pm\sqrt{\frac{1}{y} - 2}$$

$$g^{-1}(x) = \pm\sqrt{\frac{1}{x} - 2}$$

g^{-1} does not exist because for one value of x we have output of two values.
Hence it is one to many and inverse does not exist.

(c)

As $f(x) = \sqrt{x} - 1 \qquad x > 1$

$$f\left(\frac{25}{16}\right) = \sqrt{\frac{25}{16}} - 1$$

$$= \frac{5}{4} - 1$$

$$f\left(\frac{25}{16}\right) = \frac{1}{4}$$

By given conditions

$$hf(x) = f\left(\frac{25}{16}\right)$$

$$\Rightarrow \frac{1}{(\sqrt{x}-1)^2+2} = \frac{1}{4} \quad \because \text{we apply } h \text{ on } f$$

$$(\sqrt{x}-1)^2+2 = 4$$

$$(\sqrt{x})^2 - 2\sqrt{x} + 1 + 2 = 4$$

$$x - 2\sqrt{x} + 1 + 2 - 4 = 0$$

$$x - 2\sqrt{x} - 1 = 0$$

$$x - 1 = 2\sqrt{x}$$

squaring

$$(x-1)^2 = 2\sqrt{x}$$

$$x^2 - 2x + 1 = 4x$$

$$x^2 - 4x - 2x + 1 = 0$$

$$x^2 - 6x + 1 = 0$$

$$a = 1,\ b = -6,\ c = 1$$

$$x = \frac{6 \pm \sqrt{(6)^2 - 4(1)(1)}}{2 \times 1} \quad \text{By quadratic formula}$$

$$= \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 2\sqrt{8}}{2} = \frac{2(3 \pm \sqrt{8})}{2}$$

$$= 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}$$

$$x = 3 + 2\sqrt{2} \quad \because x \geq 0$$