Cambridge International AS & A Level

Mathematics 9709/32

Paper 3 Pure Mathematics 3 May/June 2024

Question No (9)

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The complex numbers z and ω are defined by $z = 1 - i$ and $\omega = -3 + 3\sqrt{3}\,i$.

(a) Express zw in the form $a + ib$, where a and b are real and in exact surd form.

b) *Express* z *and* ω *in the form* $r\,e^{i\theta}$ *, where* $r > 0$ *and* $-\pi < \theta \leq \pi$*. Give the exact values of* r *and* θ *in each case .*

(c) On an Argand diagram, the points representing ω and $z\omega$ are A and B respectively. Prove that OAB is an isosceles right-angled triangle, where O is the origin.

(d) Using your answers to part (b), prove that $\tan\frac{5\pi}{12} = \frac{\sqrt{3}\ +1}{\sqrt{3}\ -1}$

Solution:

(a)

As $z = 1 - i$ and $\omega = -3 + 3\sqrt{3}\,i$

$z\omega = (1-i)(-3+3\sqrt{3}i)$

$= -3 + 3i + 3\sqrt{3}i - 3\sqrt{3}(i)^2$ $\qquad \because i^2 = -1$

$= -3 + 3i + 3\sqrt{3}i + 3\sqrt{3}$

$z\omega = (-3+3\sqrt{3}) + (3+3\sqrt{3})i$

(b) Express z and ω in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give the exact value of r and θ in each case.

Solution

As $z = 1 - i$

$|z| = \sqrt{(1)^2 + (-1)^2}$

$= \sqrt{1+1}$

$|z| = \sqrt{2}$

$r = \sqrt{2}$

$\theta = \arg z$

$= \tan^{-1}\left(\frac{-1}{1}\right)$ $\quad\because \tan^{-1}(y/x)$

$= \tan^{-1}(-1)$

$\theta = -\pi/4$ $\quad\because -\pi < \theta \leq \pi$

now $\quad z = r e^{i\theta}$

$z = \sqrt{2}\, e^{i(-\pi/4)}$

$z = \sqrt{2}\, e^{-\pi/4 i}$

now $\quad \omega = -3 + 3\sqrt{3}i$

$\omega = x + iy$

$|\omega| = r = \sqrt{x^2 + y^2}$

$= \sqrt{(-3)^2 + (3\sqrt{3})^2}$

$= \sqrt{9 + 9(3)}$

$= \sqrt{9 + 27}$

$= \sqrt{36}$

$r = 6$

now

$\theta = \arg \omega$

$= \tan^{-1}(y/x)$

$= \tan^{-1}\left(\frac{3\sqrt{3}}{-3}\right)$

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$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$\theta = \frac{2\pi}{3}$$

so $\omega = r\, e^{i\theta}$

$$\omega = 6\, e^{i(2\pi/3)}$$

(c)

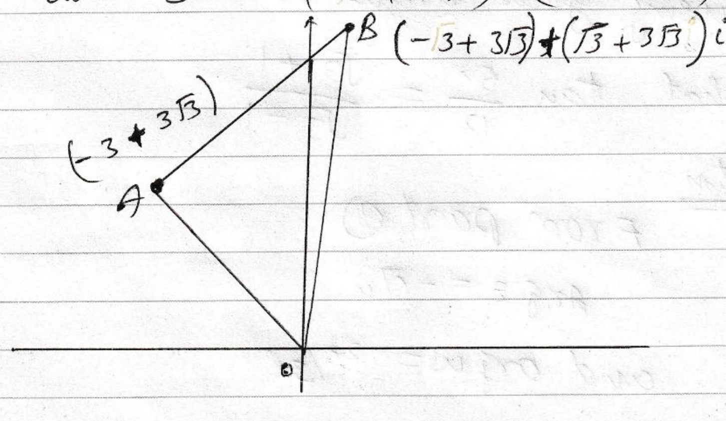
on an Argand diagram, the points representing ω and $z\omega$ are A and B respectively.

prove that OAB is an isosceles right-angled triangle, where O is the origin.

Solution

$$\text{As} \quad \omega = -3 + 3\sqrt{3}\, i$$

$$\text{and} \quad z\omega = (-\sqrt{3} + 3\sqrt{3}) + (\sqrt{3} + 3\sqrt{3})\, i \quad \text{from (a)}$$



$$|OA| = \sqrt{(-3-0)^2 + (3\sqrt{3}-0)^2}$$

$$= \sqrt{9 + 27}$$

$$|OA| = \sqrt{36} = 6$$

$$|AB| = \sqrt{(-3 + 3\sqrt{3} + 3)^2 + (3 + 3\sqrt{3} - 3\sqrt{3})^2}$$

$$= \sqrt{(3\sqrt{3})^2 + (3)^2}$$

$$= \sqrt{27 + 9}$$

$$= \sqrt{36}$$

$$|AB| = 6$$

As $|OA| = |AB| = 6$

So OAB is isosceles right-angled triangle.

(d) using your answers to part (b), prove that $\tan \frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

solution

From part (b)

$$\operatorname{Arg} z = -\pi/4$$

and $\arg w = 2\pi/3$.

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$$\arg zw = \arg z + \arg w \quad \because e^{i\theta} \cdot e^{iw} = e^{i(\theta + w)}$$

$$= -\pi/4 + 2\pi/3$$

$$= \frac{2\pi}{3} - \pi/4$$

$$\arg zw = \frac{8\pi - 3\pi}{12} = \frac{5\pi}{12} \rightarrow (1)$$

$$\text{As} \quad \arg zw = \tan^{-1}(y/x)$$

$$= \tan^{-1}\left(\frac{3+3\sqrt{3}}{-3+3\sqrt{3}}\right)$$

$$= \tan^{-1}\left(\frac{3(1+\sqrt{3})}{3(-1+\sqrt{3})}\right)$$

$$\frac{5\pi}{12} = \tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

$$\tan\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$