Cambridge International AS & A Level

Mathematics 9709

Paper 1 Pure Mathematics 1

Topic 2-Functions

Question No (21)

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The function f is defined by $f: x \to 2\,x^2 - 6x + 5 \quad for\ x\ \in \mathbb{R}$.

(i) Find the set of values of p for which the equation $f(x) = p$ has no real roots.

The function g is defined by $g: x \to 2\,x^2 - 6x + 5$ for $0 \le x \le 4$.

(ii) Express $g(x)$ in the form $a\ (x+b)^2 + c$, where a, b and c are constants.

(iii) Find the range of g.

The function h is defined by $h: x \to 2\,x^2 - 6x + 5$ for $k \le x \le 4$, where k is a constant.

(iv) State the smallest value of k for which h has an inverse.

(v) For this value of k, find an expression for $h^{-1}(x)$.

Solution

$f: x \to 2x^2 - 6x + 5$ for $x \in R$

$\Rightarrow f(x) = 2x^2 - 6x + 5$

(i) By given condition

$f(x) = p$

$\Rightarrow 2x^2 - 6x + 5 = p$ $\quad \because f(x) = 2x^2 - 6x + 5$

$2x^2 - 6x + 5 - p = 0$

comparing with

$ax^2 + bx + c = 0$ (standard quadratic equation)

$\Rightarrow a = 2,\ b = -6,\ c = 5 - p$

has no real roots

$\Rightarrow$ Discriminant, $b^2 - 4ac < 0$

$(-6)^2 - 4(2)(5-p) < 0$

$36 - 8(5-p) < 0$

$36 - 40 + 8p < 0$

$-4 + 8p < 0$

$8p < 4$

$p < 4/8$

$p < 1/2$

(ii)

$$g(x) = 2x^2 - 6x + 5$$
$$= 2(x^2 - 3x) + 5$$

completing square

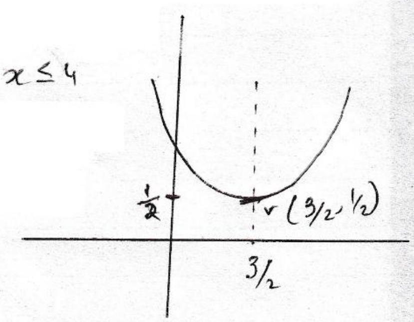
$$g(x) = 2\left(x^2 - \tfrac{2}{2}\times 3x\right) + 5$$
$$= 2\left(x^2 - 2(3/2)x\right) + 5$$
$$= 2\left(x^2 - 2(3/2)x + (3/2)^2 - (3/2)^2\right) + 5$$
$$= 2\left(x^2 - 2(3/2)x + (3/2)^2\right) - 2(3/2)^2 + 5$$
$$= 2(x - 3/2)^2 - 2 \times 9/4 + 5$$
$$= 2(x - 3/2)^2 - \frac{9}{2} + 5$$
$$= 2(x - 3/2)^2 + \frac{1}{2}$$
$$g(x) = 2\left(x + (-3/2)\right)^2 + \frac{1}{2}$$

(iii)

$$g(x) = 2x^2 - 6x + 5 \quad 0 \le x \le 4$$

at $x = 4$

$$g(4) = 2(4)^2 - 6(4) + 5$$
$$= 2(16) - 24 + 5$$
$$g(4) = 32 - 24 + 5$$
$$= 13$$



$a > 0$

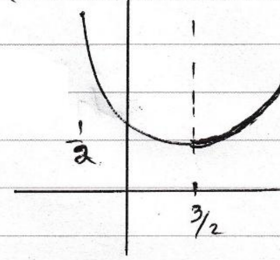
$a = 2 > 0$

so min value

$\therefore$ range of $g(x)$ is $\frac{1}{2} \le g(x) \le 13$

(iv) $h(x) = 2x^2 - 6x + 5$ for $k \le x \le 4$

For h(x) has inverse
horizontal line test
should pass, so



smallest value of k is
$k = 3/2$

(v)

$h(x) = 2x^2 - 6x + 5$

$h(x) = 2(x - 3/2)^2 + \frac{1}{2}$ part (ii)

$y = 2(x - 3/2)^2 + \frac{1}{2}$ ∴ $h(x) = y$

$y - \frac{1}{2} = 2(x - 3/2)^2$

$\frac{2y-1}{2} = 2(x - 3/2)^2$

$\frac{2y-1}{4} = (x - 3/2)^2$

$(x - 3/2)^2 = \frac{2y-1}{4}$

$x - 3/2 = \pm\sqrt{\frac{2y-1}{4}}$

$x - 3/2 = \sqrt{\frac{2y-1}{4}}$ (h(x) lies to the RHS of axis of symmetry)

$x = \sqrt{\frac{2y-1}{4}} + 3/2$

$$h^{-1}(y) = \frac{\sqrt{2y+1}}{2} + \frac{3}{2}$$

$$h^{-1}(y) = \frac{\sqrt{2y+1} + 3}{2}$$

$$h^{-1}(x) = \frac{\sqrt{2x+1} + 3}{2}$$ replacing y by x