Cambridge International AS & A Level

Mathematics 9709

Paper 1 Pure Mathematics 1

Topic 2-Functions

Question No (25)

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The function f is defined by $f: x \to 6x - x^2 - 5, \;\; x \in \mathbb{R}$,

(i) Find the set of values of x for which $f(x) \leq 3$.

(ii) Given that the line $y = mx + c$ is a tangent to the curve $y = f(x)$, show that

$$4c = m^2 - 12m + 16\,.$$

The function g is defined by $g: x \to 6x - x^2 - 5$ for x ≥ k, where k is a constant.

(iii) Express $6x - x^2 - 5$ in the form $a - (x - b)^2$, where a and b are constants.

(iv) State the smallest value of k for which g has an inverse.

(v) For this value of k, find an expression for $g^{-1}(x)$. .

Solution

$f: x \to 6x - x^2 - 5$ for $x \in R$

$f(x) = 6x - x^2 - 5$

(i)

Given conditions

$f(x) \leq 3$

$\Rightarrow 6x - x^2 - 5 \leq 3$

$6x - x^2 - 5 - 3 \leq 0$

$-x^2 + 6x - 8 \leq 0$

$-(x^2 - 6x + 8) \leq 0$

$x^2 - 6x + 8 \geq 0$

$\because a < b$

$-a > -b$

$x^2 - 4x - 2x + 8 \geq 0$

Factorization

$x(x-4) - 2(x-4) \geq 0$

$(x-4)(x-2) \geq 0$

As in $x^2 - 6x + 8$

$a = 1 > 0$

So it will have min value.

$\text{min} = \frac{4ac - b^2}{4a}$

$= \frac{4(1)(+8) - (6)^2}{4(1)}$

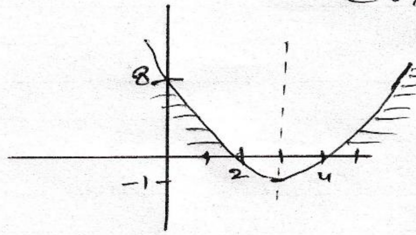
$= \frac{32 - 36}{4}$

$\text{min} = -\frac{4}{4} = -1$

Critical values

$x - 4 = 0, \; x - 2 = 0$

$\Rightarrow x = 4, \; x = 2$



As $(x-4)(x-2) \geq 0$

$x \leq 2, \; x \geq 4$

(ii) Equation of line

$$y = mx + c \longrightarrow (1)$$

Equation of curve

$$y = f(x) = 6x - x^2 - 5 \longrightarrow (2)$$

As line is tangent to the curve

$$\Rightarrow \quad mx + c = 6x - x^2 - 5$$

$$mx + c - 6x + x^2 + 5 = 0$$

$$x^2 + (m-6)x + c + 5 = 0$$

As line is tangent to the curve

$$\Rightarrow b^2 - 4ac = 0$$

$$(m-6)^2 - 4(1)(c+5) = 0$$

$$m^2 - 12m + 36 - 4(c+5) = 0$$

$$m^2 - 12m + 36 - 4c - 20 = 0$$

$$4c = m^2 - 12m + 16$$

(iii)

$$6x - x^2 - 5$$

$$= -(x^2 - 6x) - 5$$

$$= -(x^2 - 2(3)x) - 5$$

$$= -(x^2 - 2(3)(x) + (3)^2 - (3)^2) - 5$$

$= -(x-3)^2 + (3)^2 - 5$

$= -(x-3)^2 + 9 - 5$

$= -(x-3)^2 + 4$

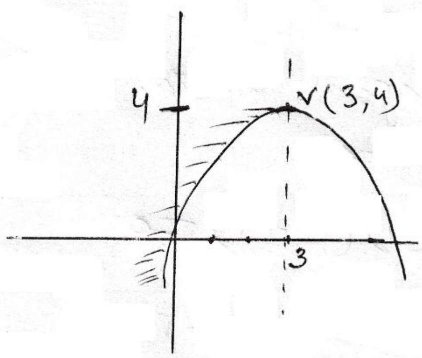
$= 4 - (x-3)^2$ required form.

(iv)

From part (iii)

$4 - (x-3)^2$

vertex $(3, 4)$



As in $4-(x-3)^2$

$a = -1 < 0$

so parabola face down.

smallest value of k for which g has an inverse is $k = 3$

(v)

As $g(x) = 4 - (x-3)^2$ from part(iii)

$\Rightarrow y = 4 - (x-3)^2 \quad \because y = g(x)$

$(x-3)^2 = 4 - y$

$x-3 = \pm\sqrt{4-y}$

$x-3 = +\sqrt{4-y}$ $\because$ symmetrical line on +ve of x-axis.

$x = \sqrt{4-y} + 3$

$g^{-1}(y) = \sqrt{4-y} + 3$ $\because x = g^{-1}(y)$

replacing y by x

$g^{-1}(x) = \sqrt{4-x} + 3$